PHYS5150 — PLASMA PHYSICS

LECTURE 10 - SLOW E FIELD VARIATIONS, POLARIZATION DRIFT, AND PLASMAS AS A DIELECTRIC

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1 POLARIZATION DRIFT

Let us return to the case of uniform orthogonal electric and magnetic fields which we have studied in lecture 6. This time we allow for slow variations of the electric field. Like in lecture 2 we start with the particle's equation of motion, but this time we take the crossproduct with $\frac{\mathbf{B}}{B^2}$

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \quad \Big| \times \frac{\mathbf{B}}{B^2},$$

and

Remember that $(A\times B)\times C = (A\cdot C)B-(B\cdot C)A$

$$\begin{aligned}
\frac{m}{q}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\times\frac{\mathbf{B}}{B^2} &= \frac{\mathbf{E}\times\mathbf{B}}{B^2} + (\mathbf{v}\times\mathbf{B})\times\frac{\mathbf{B}}{B^2} \\
\frac{m}{q}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\times\frac{\mathbf{B}}{B^2} &= \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{B}}{B^2}(\mathbf{v}\cdot\mathbf{B}) - \mathbf{v}
\end{aligned}$$

or after rearranging and using that $\dot{\mathbf{B}} = 0$

$$\underbrace{\mathbf{v}-\frac{\mathbf{B}}{B^2}(\mathbf{v}\cdot\mathbf{B})}_{\perp \text{ velocity}} = \underbrace{\frac{\mathbf{E}\times\mathbf{B}}{B^2}}_{\mathbf{E}\times\mathbf{B}\text{ drift}} - \frac{m}{q}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\times\frac{\mathbf{B}}{B^2} = \mathbf{v}_E - \frac{m}{qB^2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\times\mathbf{B}).$$

The left side can be interpreted as a perpendicular velocity and the first term on the right side is again the $\mathbf{E}\times\mathbf{B}$ drift. Now we average over one gyroperiod

$$\mathbf{v}_d = \mathbf{v}_E - \frac{m}{qB^2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\times\mathbf{B})$$

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and identify the left side as a drift velocity. In lecture 6 we have found that the $\mathbf{E} \times \mathbf{B}$ drift results from a Lorentz transformation $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$ into the particle's moving reference frame and the fact that $\mathbf{E}'$ must vanish for a free particle. Similarily, the $\mathbf{v} \times \mathbf{B}$ term represents a perpendicular electric field $\mathbf{E}_\perp = -\mathbf{v} \times \mathbf{B}$, and thus

$$\mathbf{v}_d = \mathbf{v}_E + \frac{m}{qB^2}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t} = \mathbf{v}_E + \frac{q}{|q|}\frac{1}{\omega_c B}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t}.$$

The last term is called *Polarization drift*

$$\boxed{\mathbf{v}_p = \frac{q}{|q|}\frac{1}{\omega_c B}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t},}$$

which results from a slow variation of the electric field. $\mathbf{v}_p$ is proportional to the particle mass and depends on its polarity. Thus, the polarization drift creates a current into the direction of the electric field

$$\mathbf{j}_p = n_0 e(\mathbf{v}_{pi} - \mathbf{v}_{pe}) = \frac{n_0(m_e + m_i)}{B^2}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t}.$$

2 DIELECTRIC CONSTANT OF A PLASMA

A dielectric material is an insulator which gets polarized by an electric field. Clearly we expect such a behavior from a plasma. To see this, lets position a plasma between the plates of a capacitor

$$C = (\epsilon_0 + \epsilon_p)\frac{A}{d},$$

where A is the area of the plates separated by d. $\epsilon_0 = 8.854\,\mathrm{F\,m^{-1}}$ is the permittivity of the vacuum, and ϵ_p is the permittivity of the plasma we are interested in. When applying an alternating voltage V at the capacitor, we will observe the current

$$i = C\frac{\partial V}{\partial t},$$

which after substituting C into it and expressing V/d by E

$$i_p = \epsilon_p A \frac{1}{d}\frac{\partial V}{\partial t} = \epsilon_p A \frac{\partial E}{\partial t}.$$

On the other hand, in the previous section we have just found the current resulting from an alternating electric field applied to a plasma

$$i_p = j_p A = \frac{\rho_m}{B^2} A \frac{\partial E}{\partial t}.$$

Comparing the two expressions for i_p yields the *low frequency plasma permittivity for transverse motion*

$$\epsilon_p = \frac{\rho_m}{B^2},$$

where ρ_m is the plasma's mass density. We get further insight into the nature of ϵ_p by substituting characteristic plasma parameters into the relative permittivity

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = k = 1 + \frac{\rho_m}{\epsilon_0 B^2},$$

where ϵ is the total permittivity

$$\epsilon = \epsilon_0 + \frac{\rho_m}{B^2} = \epsilon_0 \left(1 + \frac{\rho_m}{\epsilon_0 B^2}\right).$$

Now,

$$\begin{aligned}
\epsilon_r &\approx 1 + \frac{n m_i}{\epsilon_0 B^2} = 1 + \frac{e^2}{e^2}\frac{m_e}{m_e}\frac{n m_i}{\epsilon_0 B^2} \\
&\approx 1 + \underbrace{\left(\frac{n e^2}{\epsilon_0 m_e}\right)}_{\omega_p^2} \underbrace{\left(\frac{m_i}{eB}\right)}_{1/\omega_{c,i}} \underbrace{\left(\frac{m_e}{eB}\right)}_{1/\omega_{c,e}}
\end{aligned}$$

and hence,

$$\boxed{\frac{\epsilon}{\epsilon_0} = \frac{\omega_p^2}{\omega_{c,i}\omega_{c,e}}.} \tag{1}$$

The low frequency plasma permittivity depends only on the plasma frequency and the cyclotron frequencies of the ions and electrons. We will derive later the same expression more rigorously using the fluid description of the plasma. This will also elucidate the meaning of the simplifications we have made to obtain Eq. (1).

3 WHERE DO WE STAND?

Parameters	Drifts	Drift currents
$\rho_c = \frac{m v_\perp}{\lvert q\rvert B}$	$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2}$	
$\omega_c = \frac{\lvert q\rvert B}{m}$	$\mathbf{v}_p = \frac{1}{\omega_p B}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t}$	$j_p = \frac{n_e(m_i+m_e)}{B^2}\frac{\mathrm{d}\mathbf{E}_\perp}{\mathrm{d}t}$
$\mu = \frac{T_\perp}{B}$	$\mathbf{v}_G = \frac{T_\perp}{qB}\left[\frac{\hat{\mathbf{B}}\times\nabla\mathbf{B}}{B}\right]$	$j_G = \frac{n_e(m_i+m_e)}{B^2}(\mathbf{B}\times\nabla\mathbf{B})$
	$\mathbf{v}_c = \frac{2T_\parallel}{qB}\left[\frac{\hat{\mathbf{B}}\times\hat{\mathbf{R}}_c}{R_c}\right]$	$j_R = \frac{n_e(T_\parallel^i+T_\parallel^e)}{B^2R_c^2}(\mathbf{B}\times\mathbf{R}_c)$

INVARIANTS

1. μ

2. $J = m \int \mathbf{v}_{\|} \, \mathrm{d}s$

3. $\Phi = \oint v_d r \, \mathrm{d}\phi = \frac{2\pi m}{q^2} M$

4 FLUID DESCRIPTION OF PLASMAS

4.1 *Fluid variables*

We first define the volume of a fluid parcel as

$$\mathrm{d}x^3 = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

which has the mass

$$m = (n_i m_i + n_e m_e) \, \mathrm{d}x^3$$

and the mass density

$$\rho = \frac{m}{\mathrm{d}x^3} = (n_i m_i + n_e m_e) = \rho_i + \rho_e.$$

Mass m and density ρ are fluid variables. We now introduce the average velocity $\mathbf{u}$ of the plasma particles in the fluid parcel. Obviously, the fluid parcel will flow at this speed. The distribution $f(\mathbf{v}|\mathbf{u})$ of particle speeds can often described by a shifted Maxwellian, i.e.

$$f(\mathbf{v}|\mathbf{u}) = \frac{n}{(\pi v_{th}^2)^{3/2}} \exp\left\{ -\frac{(\mathbf{v}-\mathbf{u})^2}{v_{th}^2} \right\}.$$

Knowledge of $f(\mathbf{v})$ allows us to compute $\mathbf{u}$

$$\mathbf{u} = \frac{\int \mathbf{v} f(\mathbf{v}) \, \mathrm{d}x^3}{\int f(\mathbf{v}) \, \mathrm{d}x^3} = \frac{1}{n} \int \mathbf{v} f(\mathbf{v}) \, \mathrm{d}x^3.$$

The next fluid variable to discuss is the particle flux through a face of the fluid volume $\mathrm{d}x^3$. Let us consider the flux through the $\mathrm{d}x \, \mathrm{d}y$ face. Because flux is number of particles per area and time we can write

$$\Gamma_{xy} = \frac{N}{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t},$$

or after multiplying with $\mathrm{d}z / \mathrm{d}z$

$$\Gamma_{xy} = \frac{N}{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z} \frac{\mathrm{d}z}{\mathrm{d}t} = n u_z,$$

where u_z is the z component of fluid velocity. In general

$$\bar{\Gamma} = n\mathbf{u}.$$

The current density $\mathbf{j}$ of the fluid parcel is the charge flux

$$\mathbf{j} = nq\mathbf{u}.$$

The last fluid variable we need to consider is the pressure, which in fact is the flux of momentum evaluated in the frame moving with $\mathbf{u}$.

4.2 *Continuity equation*

$$\frac{\partial}{\partial t} m = \frac{\partial}{\partial t}(\rho\, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z) = \sum \text{inward-flow} \;\; = \rho u_x\,\mathrm{d}y\,\mathrm{d}z$$

6 sides:

$$\int \frac{\partial \rho}{\partial t}\,\mathrm{d}x^3 = -\oint \rho\mathbf{u}\,\mathrm{d}A$$

divergence theorem: $\oint \mathbf{F}\,\mathrm{d}A = \int (\nabla\mathbf{F})\,\mathrm{d}x^3$:

$$\int \frac{\partial \rho}{\partial t}\,\mathrm{d}x^3 = -\int \nabla(\rho\mathbf{u})\,\mathrm{d}x^3$$

or

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla(\rho\mathbf{u})\right]\,\mathrm{d}x^3 = 0$$

Continuity equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla(\rho\mathbf{u}) = 0}$$

In general, if $\mathbf{H}$ is conserved, then

$$\frac{\partial}{\partial t}\mathbf{H} + \nabla(\mathbf{H}\mathbf{u}) = 0,$$

where $(\mathbf{H}\mathbf{u})$ is the flux of $\mathbf{H}$. If there is a source term $S = \frac{\text{new mass}}{\mathrm{d}x^3}$, then we need to change the continuity equation to

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla(\rho\mathbf{u}) = S.}$$